Controlling simple dynamics by a disagreement function

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We introduce a formula for the disagreement function which is used to control a recently proposed dynamics of the Ising spin system. This leads to four different phases of the Ising spin chain at zero temperature. One of these phases is doubly degenerated (antiferromagnetic and ferromagnetic states are equally probable). On the borders between the phases two types of transitions are observed: infinite degeneration and instability lines. The relaxation of the system depends strongly on the phase.

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I. INTRODUCTION

The Ising spin system is one of the most frequently used models of statistical mechanics. Its simplicity (binary variables) makes it appealing to researchers from other branches of science including biology [1], sociology [2], and economy [3,4]. In sociophysics models of opinion formation based on the social impact theory (reviewed in [5]), the individual opinion is described by the Ising spin. This corresponds not only to typical "yes"-"no" questions, but also to important issues where the distribution of opinion seems to be bimodal, peaked on extreme values. In general, in these models the influence flows inward from the border to the center, like in the majority rules, where the site in the middle takes the state of the majority of neighboring sites. In contrast, in USDF (an abbreviation from the sociological rule "United we stand, divided we fall") model [6] an outward flow of influence is imposed. In the USDF model, an isolated person does not convince others; however, a group of people sharing the same opinion influences their neighbors. In spite of simple rules the model exhibited complicated dynamics in one [6] and more dimensions (reviewed in [7]). In less than a year, this model has found several applications: e.g., it was used to explain the distribution of votes among candidates in Brazilian local election [8] and to model the price dynamics of financial instruments [9].

In this paper we introduce the *"disagreement function"* [3] which is used to control the dynamics of the model. We show that for a one-dimensional Ising spin chain at zero temperature this leads to four different phases: ferromagnetic, antiferromagnetic, (2,2) antiphase, and a doubly degenerated phase in which both the ferromagnet and antiferromagnet phases are equally probable stable steady states of the system. Apart from structural differences between phases the difference in relaxation will be shown. The system in general will relax in two different ways depending on the phase. Moreover, a sharp change of the relaxation time on borders of the phases will be observed.

II. THE MODEL

Recently a simple model for opinion evolution in a closed community was proposed [6]. In this model the community is represented by a horizontal chain of Ising spins, which are either up or down. A pair of parallel neighbors forces its two neighbors to have the same orientation (in random sequential updating), while for an antiparallel pair, the left neighbor takes the orientation of the right part of the pair, and the left neighbor follows the right part of the pair. Thus the model can be described by two simple dynamic rules:

(i) D_1: $S_{i-1}(t+1)=S_i(t)$ and $S_{i+2}(t+1)=S_i(t)$ if $S_i(t)*S_{i+1}(t)=1$.

(ii) D_2: $S_{i-1}(t+1)=S_{i+1}(t)$ and $S_{i+2}(t+1)=S_i(t)$ if $S_i(t)*S_{i+1}(t)=-1$.

In contrast to the usual majority rules [10], in this model the influence does not flow inward from the surrounding neighbors to the center site, but spreads outward from the center to the neighbors. The model thus describes the spread of opinions. The dynamic rules lead to two different stable steady states (ferromagnetic and antiferromagnetic) with equal probability. The second dynamic rule (D_2) of the model has been already changed in two different ways. In the case of antiparallel spins the neighboring spins can either flip with probability 1/2 [9] (D_{2A}) or remain unchanged [7] (D_{2B}). In both cases (D_{2A} and D_{2B}) the only final state is ferromagnet. It is worth mentioning that the ferromagnetic state for both rules, D_{2A} and D_{2B}, is always reached (even in two dimensions) in contrast to the Ising spin system under Glauber dynamics [11,12]. In the case of D_{2B} besides of ferromagnetic stable steady states, the antiferromagnetic unstable steady state exists.

Since we have up till now three different rules for the case of antiparallel spins, we propose a generalization of the previous models. The generalized model consists of two components (TC) hence the name TC model.

(i) The dynamics: choose a pair of spins S_{i+1} and S_{i+2} and change its next nearest neighbors S_i and S_{i+3}.

(ii) The rules: control the dynamics of the ith and $(i+3)$th spins by the disagreement function.

In the following sections we introduce the disagreement function and show that the TC model includes as special cases all earlier proposed models [6,7,9]. Moreover, the TC model consists of more than those three subcases which we present on its phase diagram. Using Monte Carlo simulations we show how the system described by the TC model relaxes.

III. HOW TO CONTROL DYNAMICS?

Let us assume for a while that we have the formula for a function that can control TC dynamics and denote it by E.

We choose at random a pair of spins S_{i+1} and S_{i+2} and we calculate $E^{+}=E(S_i,S_{i+1},S_{i+2})$. Next we calculate $E^{-}=E(-S_i,S_{i+1},S_{i+2})$ in the case of flipped ith spin. If $E^{-}<E^{+}$ then we will flip the ith spin; if not, the spin will remain unchanged. We do the same for the second neighbor of the chosen pair, i.e., for the spin S_{i+3}.

Our dynamics looks now similar to the Glauber dynamics in zero temperature, where E plays the role of energy. However, there are three main differences between these two dynamics.

(i) In the Glauber dynamics we flip the ith spin according to the interactions with the $(i-1)$th and $(i+1)$th spins; here we look at the $(i+1)$th and $(i+2)$th spins.

(ii) In the Glauber dynamics the flip can be done even if the old energy is equal to the new one. In my opinion flipping a spin without any loss of the energy at $T=0$ is not very natural, but is needed to get the ground state (in two dimensions even this is not enough [12]).

(iii) In our case, E is called the disagreement function, since it is not the energy. On the contrary, the Glauber dynamics deals with the real energy (i.e., the sum of interactions with all neighbors).

Now we will look for the formula for E. We shall deal with the lattice model where each lattice site i is occupied by an Ising spin $S_i=\pm 1$. Usually, the spins are assumed to interact through pairwise coupling of the form $-J_{ij}S_iS_j$, where J_{ij} are exchange integrals. Of course, the ordering of the spins is determined by the interactions. One of the best studied examples is the nearest neighbor (NN) Ising model with ferromagnetic coupling, i.e., $J_{ij}=J>0$ for neighbor spins S_i and S_j, while $J_{ij}=0$ for more distant spins. Certainly, in a such model, the spins form the ferromagnetic state (all spins up or all spins down) at the ground state. For $J<0$ the antiferromagnetic state is formed at $T=0$.

In the TC model the ith spin interacts with its two neighbors, and the one-dimensional (1D) Hamiltonian can be written in the following form:

$$H=-J_1\sum_i S_iS_{i+1}-J_2\sum_i S_iS_{i+2}. \quad (1)$$

For $J_1>0$ and $J_2<0$ this is the well known ANNNI (axial next-nearest-neighbor Ising) model introduced in [13] and reviewed in [14]. It describes the Ising spin chain with ferromagnetic interaction $J_1>0$ between nearest neighbors (NN) and antiferromagnetic interactions between next nearest neighbors (NNN). Of course, in the one-dimensional case truly ordered states are stable only at zero temperature $T=0$. If we introduce the competition ratio $r=-J_2/J_1$ we get in $T=0$ ferromagnetic state for $r<1/2$ and (2,2) structure for $r>1/2$. Interestingly, for all $r<0$, the equilibrium ground state cannot be reached via single spin-flip Glauber dynamics [15]. In contrary in TC model single spin-flip is sufficient to get the ground steady state.

Now, we will use the NNN Ising Hamiltonian [1] to construct the disagreement function E. Chowdbury and Stauffer introduced similarly a disagreement function based on the simple NN Ising hamiltonian to the model of financial market [3]. We write E in the following form:

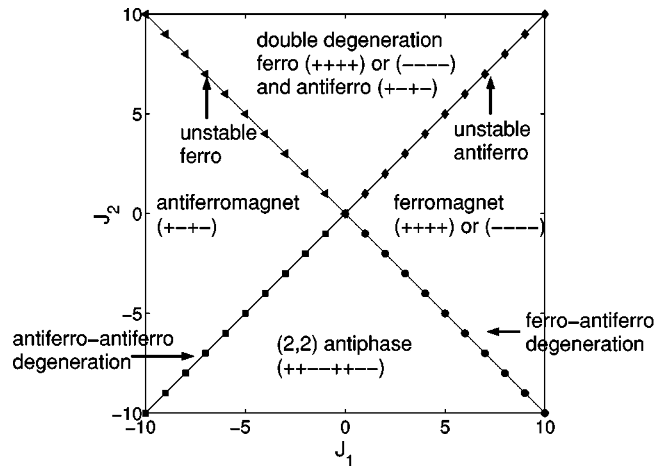

FIG. 1. The phase diagram of the TC model.

$$E=-J_1S_iS_{i+1}-J_2S_iS_{i+2}. \quad (2)$$

Each individual would like to minimize the corresponding disagreement function. In the TC dynamics we choose a pair S_{i+1} and S_{i+2} and we change its neighbor S_i (we also change S_{i+3} spin calculating $E=-J_1S_{i+3}S_{i+2}-J_2S_{i+3}S_{i+1}$, but for simplicity we further write only about the ith spin). For these three spins (S_i,S_{i+1},S_{i+2}) we have four values of E:

(1) $+++$ or $---$ gives $E_1=-(J_1+J_2)$.
(2) $-++$ or $+--$ gives $E_2=J_2-J_1$.
(3) $+-+$ or $-+-$ gives $E_3=J_1-J_2$.
(4) $--+$ or $++-$ gives $E_4=J_1+J_2$.

It is worth noticing that the possible transitions are only between states 1 and 2 or between 3 and 4. Now we can derive from the TC model all previous models.

(a) We have the USDF model [6] if $S_{i+1}(t)*S_{i+2}(t)=1$ then $S_i(t+1)=S_{i+1}(t)$, i.e., $E_1<E_2$ if $S_{i+1}(t)*S_{i+2}(t)=-1$ then $S_i(t+1)=S_{i+2}(t)$, i.e., $E_3<E_4$. Thus the USDF model corresponds to the TC model with $-J_2<J_1<J_2$.

(b) We have the model of the financial market [9] if $S_{i+1}(t)*S_{i+2}(t)=1$ then $S_i(t+1)=S_{i+1}(t)$ if $S_{i+1}(t)*S_{i+2}(t)=-1$ then $S_i(t+1)=-S_i(t)$ with probability 1/2. This corresponds to the TC model with $E_1<E_2$ and $E_4<E_3 \Rightarrow -J_2<J_1$ and $J_1>J_2$.

(c) Other models reviewed in [7] are as follows: if $S_{i+1}(t)*S_{i+2}(t)=1$ then $S_i(t+1)=S_{i+1}(t)$, i.e., $E_1<E_2$ if $S_{i+1}(t)*S_{i+2}(t)=-1$ then $S_i(t+1)=S_i(t)$, i.e., $E_3=E_4$. These models correspond to the TC model with $J_1=J_2$.

There are of course more subcases of the TC model depending on the interaction coefficients J_1 and J_2. In Fig. 1 all possible phases, depending on the interaction coefficients, are presented. The North (doubly degenerated) phase corresponds to the original rule D_2. The East (ferromagnetic) phase corresponds to rule D_{2A} (the flip in case of antiparallel spins is made at random). The line between these two phases corresponds to rule D_{2B} (the flip is possible only in the case of parallel spins). On this line the antiferromagnetic steady state still exists but it becomes unstable and we never reach it outside of this state. It is also interesting to see what happens on other border lines.

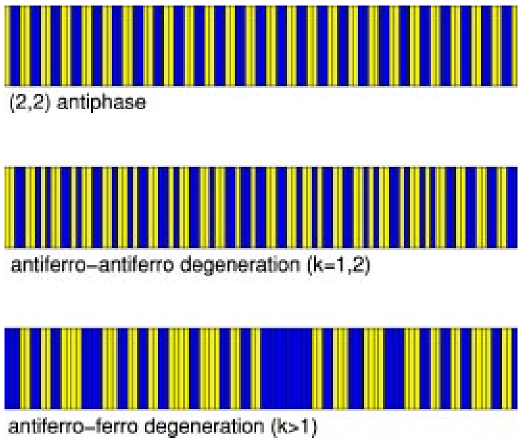

FIG. 2. Examples of three different steady states of the TC model are presented. Bright lines denote spins up and dark lines denote spins down.

The border between the ferromagnetic state and the (2,2) antiphase (see Fig. 2) is infinitely degenerated. Let us define (after [14]) a k band formed by k adjacent, identically oriented spins, terminated at the both ends by opposite oriented spins. With such a definition, the ferromagnetic structure is a zero-band structure, the antiferromagnetic phase is a one-band structure and the (2,2) antiphase is a two-band structure. On the line between the ferromagnet and (2,2) antiphase any sequence of k band ($k \geqslant 2$) is equally probable (see Fig. 2). The line between the (2,2) antiphase and the antiferromagnet is also degenerated, and any sequence of k band (with $k=1,2$) is the steady state (see Fig. 2).

There is also another interesting feature which differs phases from each other—the time and the style in which the system relaxes. We will describe it in the next section.

IV. HOW DOES THE SYSTEM RELAX?

What happens when we suddenly cool our system from a high temperature to zero temperature? As we mentioned previously the system will relax to one of the possible final states described by the phase diagram (Fig. 1). But how does it relax? We studied this using Monte Carlo simulations. We found out that the relaxation process strongly depends on the phase. The system can reach antiferromagnetic state in the West (antiferromagnetic) phase as well as in the North (degenerated) phase. However, it will relax to this state differently in each case. In the antiferromagnetic phase [Fig. 3(c)] the system will be almost totally ordered after several Monte Carlo Steps (MCS). Then the system will oscillate around the final state. These oscillations will decrease in time and finally the system will reach the steady state. In the degenerated phase [Fig. 3(a)] the system will order very slowly.

In Fig. 3 the examples of relaxations in all four phases are presented. To show this relaxation we choose the opinion changes, since the model was proposed to investigate the opinion dynamics. We defined the opinion [6] as a magnetization of the system:

$$m=\sum_{i=1}^{N} S_i . \tag{3}$$

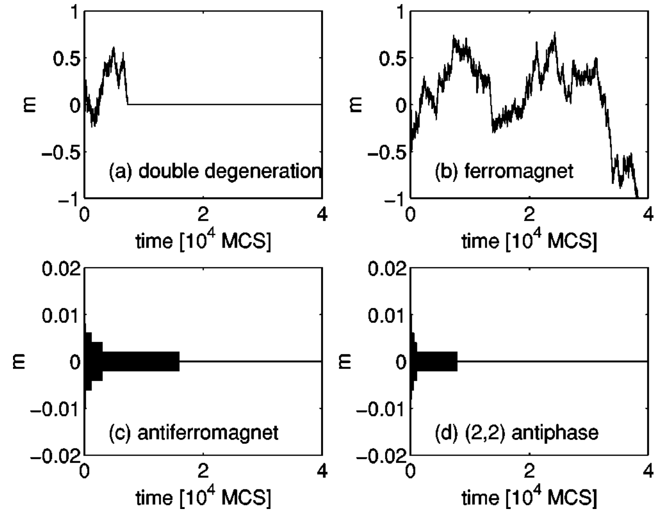

FIG. 3. Examples of the relaxation for 1000 spins system are presented. Two kinds of relaxations were observed depending on interaction coefficients. For $J_1 > -J_2$ the system makes a long "random" walk to the final state, while for $J_1 < -J_2$ the system makes decreasing oscillations around the final state.

For such a choice the system will relax to $|m|=1$ (ferromagnet) or $|m|=0$ [antiferromagnet or (2,2) antiphase]. Of course, one could also choose the two point correlation function $g=\langle S_i S_{i+1}\rangle$ to see how the system relaxes. We have done it to recognize the final state ($g=1,-1$ or 0 for the ferromagnet phase, antiferromagnet phase and the (2,2) antiphase, respectively). For $J_1 > -J_2$ (North and East parts of the diagram in Fig. 1) the ordering of the system is very slow. Sometimes the opinion can change dramaticaly in a short time (see Fig. 3). The long time trends are observed, which reminds us very much of the real sociological process [6]. For $J_1 < -J_2$ the system is almost ordered after several Monte Carlo steps; however, then it takes a long time to reach the real final steady state. The opinion is fluctuating around zero and these fluctuations are decreasing in time (see Fig. 3). Although the way in which the system relaxes in the North and East phases is the same, the relaxation time in each of these phases is different. About a two times shorter (on average) time is needed to reach the final state in the degenerated phase. The relaxation time changes very sharply on the border between these two phases (Fig. 4). A similar effect is observed also on the border between the antiferromagnetic and degenerated phases.

Let us now understand more deeply the relaxation of the system. We first focus on the case with $J_1 \in (J_2, -J_2)$, for which the (2,2) antiphase is the ground state (the South part of the phase diagram in Fig. 1), as this case leads to an interesting dynamics. We will follow the way in which it was done for the one-dimensional ANNNI model under Glauber dynamics [15]. Starting for simplicity from an initial ferromagnetically ordered up state, this system evolves to the state $[\ldots ++--++--++\ldots]$. In the TC model we choose a pair of spins at random and we try to flip its neighbors. The disagreement function decreases by $\Delta E=4(J_1+J_2)$ when two NN spins flip to create two isolated down spins. The same E loss arises for any nucleation event which

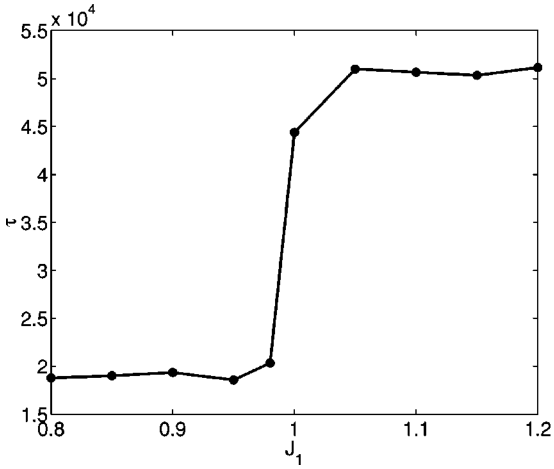

FIG. 4. Relaxation time for $J_2=1$. In this figure we present results for the system of 1000 spins averaged over 10 000 samples.

occurs within the domain of length $\geqslant 4$. After this nucleation, the single spin domain can grow to length 2, decreasing the disagreement function by $\Delta E = 2(J_2 - J_1)$,

$$+ + + + + + + + + +$$

$$\Downarrow$$

$$+ + + - + + - + + +$$

$$\Downarrow$$

$$+ + + - + + - - + +$$

$$\Downarrow$$

$$+ + - - + + - - + + .$$

At the end of this nucleation stage, the system consists of ordered (2,2)-antiphase regions as well as domains of size 1, 3, and 4. These remaining domains now undergo a sequence of reactions which lead the system to the ground state. 1-domains and 3-domains diffuse freely within a sea of 2-domains, analogously like in the one-dimensional ANNNI model under the Glauber dynamics [15]. Interestingly, the 4-domain behaves more complicated for the TC model then for the ANNNI model with Glauber dynamics. In the latter case 4-domain splits into two 3-domains,

$$4 \to 3 + 3$$

$$- - + + - - - - + + - - + +$$

$$\Downarrow$$

$$- - + + - - - + + + - - + + .$$

On the other hand, a 4-domain will form when two 3-domains collide.

In the case of the TC model there are two possibilities,

$$4 \to 3 + 3$$

$$- - + + - - - - + + - - + +$$

$$\Downarrow$$

$$- - + + - - - + + + - - + +$$

or

$$4 \to 1 + 1$$

$$- - + + - - - - + + - - + +$$

$$\Downarrow$$

$$- - + + - + - - + + - - + + .$$

These 1- and 3-domains can create again a 4-domain or diffuse freely in the system. Now we can ask what is the process which leads finally to the ground state. Imagine that a 1-domain meets a 3-domain. They annihilate to form a stable 2-domain, like in the case of the ANNNI model under Glauber dynamics [15]:

$$1 + 3 \to 0$$

$$+ + - - + - - - + + - -$$

$$\Downarrow$$

$$+ + - - + + - - + + - - .$$

Since each of the described processes leads to a loss of E, they each occur at the same rate when $T=0$. Thus, while the nucleation process, which leads to an almost ordered state, is very fast in the TC model, the second step which leads to the final state is rather slow. The magnetization is almost zero after the first nucleation step (which takes several MCS). In the second step it oscillates because of diffusive domains and decreases due to annihilation processes.

Now we look quickly at the case of the ferromagnetic state (the East part of the phase diagram in Fig. 1). In this case, in the first step small domains are created and then they grow slowly. The reaction is possible only when the pair which changes its neighbors (underlined) touches a wall of a domain:

$$+ + + - - - - \underline{- -} + + + +$$

$$\Downarrow$$

$$+ + + - - - - - - + + + .$$

If we choose a pair inside the domain (and not touching the wall of this domain) or on the border between domains nothing will happen, thus the relaxation is slow. Domain walls follow a random walk resulting the process with long up and down trends (see Fig. 3).

In the case of double degeneration (the North part of the phase diagram, Fig. 1) the relaxation is similar but faster. In this case reaction takes place not only in the neighborhood of domain walls (in this case a local ferromagnet is created), but also on the border between domains (in this case a local antiferromagnet is created):

$$+++----\underline{-+}+++$$

$$\Downarrow$$

$$+++---+-+-++.$$

If d denotes the number of domains then for the ferromagnetic case there are $2d$ points where the reaction can take place and for the degenerated phase there are $3d$ reaction points. Apart from more reaction points there is another reason for which relaxation is slower in the ferromagnetic phase. In this case for each chosen pair we change at most one spin. In the degenerated phase we can change two spins if the chosen pair is on the border of domains (look at the above example). This explains the difference between relaxation times in both phases shown in Fig. 4.

V. SUMMARY

We proposed a new generalized model of opinion formation. The disagreement function was introduced to control the simple dynamics of an Ising spin chain at zero temperature. This allowed us to generalize the previous model of opinion dynamics. It was shown that the phase diagram for that system described by such a model consists of four different phases. The most interesting is the existence of the doubly degenerated phase in which the system can reach the antiferromagnetic steady state or the ferromagnetic steady state with the same probability. Moreover, it was shown that the system can relax in two different ways depending on the interaction coefficients. Surprisingly the system can reach the antiferromagnetic state in two different ways. In the antiferromagnetic phase the system will be almost ordered after several Monte Carlo steps and then decreasing oscillations around the final state will lead the system into this state. In the degenerated phase, the system will behave "blindly" making a long "random" walk to the final state. It would probably be worth looking at the system described by such a model in higher dimensions and higher temperature. We also hope that the generalized TC model will find as many applications as its older brothers [6].

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